The Harder Narasimhan Filtration and the Schatz Polygon

IDC451 Presentation

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Let M be a Riemann surface which is a complex manifold of complex dimension 1. Then we have the following definition of a vector bundle of rank r .

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2. There exists a neighborhood U and a homeomorphism ψ_U such that the following commutative diagram holds

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\psi_U} & U \times \mathbb{C}^r \\ & {\searrow}^{p} \quad {\swarrow}_{\mathrm{pr}} & \\ & U & \end{array}$$

3. $\psi_V \circ \psi_U$ is of the form $(z, w) \mapsto (z, A(z)w)$ where $A : U \cap V \to GL(m, \mathbb{C})$ is a holomorphic map

Examples

Example (Line Bundles)

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Example (Product bundle)

Let M be a Riemann surface, then we have $\pi : M \times \mathbb{C}^r \to M$ where π is the projection to the first factor M. This is called the trivial bundle and the isomorophism is called a global trivialisation.

Examples

Example (Determinant Line bundle)

Given a vector bundle E of rank r, the r-th exterior product of E is defined as the determinant line bundle. It is deonoted as $\det(E) = \wedge^r E$

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The transition maps of the determinant line bundle are $\wedge^r g_{UV} : U \cap V \to GL(\wedge^r \mathbb{C}^r = \mathbb{C}^*)$ where g_{UV} are the transition maps of E. One can explicitly write the maps as follows

$$(\wedge^r g_{UV})(e_1 \wedge \cdots \wedge e_r) = \wedge_{k=1}^r g_{UV}(e_k) = \wedge_{i=1}^r \sum_{k=1}^n g_{ki}(e_k)$$

The isomorphism class of line bundles on a Riemann surface M in fact form a group, known as the Picard Group, often denoted as $\mathrm{Pic}(X)$. An important invariant to classify $\mathrm{Pic}(X)$ is known as the degree, which assigns an integer d to every line bundle $L \in \mathrm{Pic}(X)$. An explicit construction of a degree of a vector bundle in **[1]**.

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The degree of a vector bundle E of rank r is defined as the degree of the determinant line bundle $deg(E) = deg(\det E)$ as above

The above two notions degree and rank turn out to be extremely powerful in the scenario of constructing moduli space of vector bundles. We proceed now with the definition of slope stability.

Definition (Slope)

Let M be a Riemann surface. The slope of a non-zero vector bundle E on M is the rational number

$$\mu(E) := \frac{deg(E)}{rk(E)} \in \mathbb{Q}$$

Stability

Definition

A vector bundle is called *slope stable* if for all non-trivial sub bundles $F \subset E$ one has

$$\mu(F) < \mu(E)$$

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An intuitive understanding of the notion of slope would be if there exist a nonzero morphism between two vector bundles $\phi : E \to E'$ then $\mu(E) \geq \mu(E')$

Theorem

A vector bundle E on M is stable (semi-stable) iff $\mu(E/F) > (\geq)\ \mu(E)$ for all such F

A filtration for vector bundles

In the case of ring theory and module theory, obtaining a filtration for rings/modules (for example in the case of Noetherian rings) is naturally available through the definitions of these objects. However in the case of vector bundles there is no such canonical filtrations available. Harder and Narasimhan in **[2]** showed that such a filtration maybe obtained through stable bundles (defined more than a decade ago by Mumford).

A filtration for vector bundles

Theorem

Let E be a vector bundle over a riemann surface M. Then there exists a unique filtration by sub-bundles such that

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that the following conditions holds

- $\forall i \in \{1; \ldots ; l\}$ *the vector bundle* E_i/E_{i-1} *is semistable*
- *The slopes of the successive quotients satisfy*

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_l/E_{l-1})$$

Let us set $r = rk(E)$, $d = deg(E)$ and denote $r_i = rk(E_i/E_{i-1})$, $d_i = deg(E_i/E_{i-1})$ Since rank and degree is additive on short exact sequences we have

$$r_1 + \cdots + r_l = r \text{ and } d_l + \cdots + d_l = d$$

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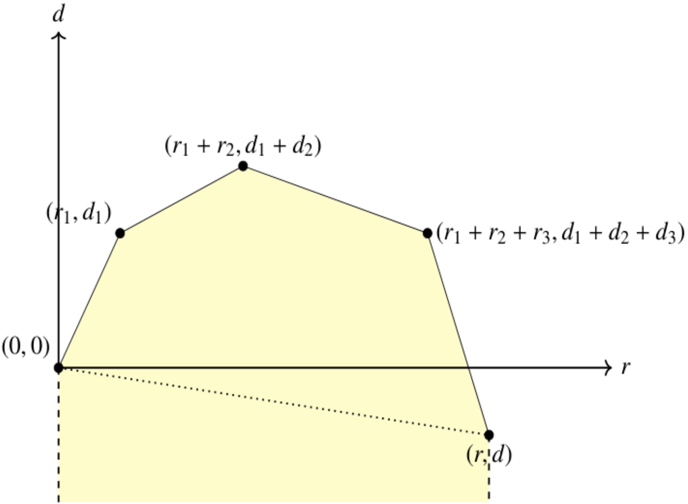
Consider the polygonal line defined by the points in the plane (r, d) :

$P_\mu = \{(0,0), (r_1, d_1), (r_1+r_2, d_1+d_2), \ldots, (r_1+\cdots+r_l, d_1+\cdots+d_l)\}$



[1]

[1] Credit: Vector bundles over Algebraic curves, F. Schaffhauser, 2023

In the above picture, (which is for $l = 4$) the slope of the line segment going from (r_1, d_1) to $(r_1 + r_2, d_1 + d_2)$ is $d_2/r_2 = \mu(E_2/E_1)$, i.e. it is equal to the slope of the vector bundle E_2/E_1 and likewise for higher indices.

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Note that the above polygon is a line if and only if E is semistable i.e, $0 \subset E$. In other words, everything located above that line measures the defect of semistability of the bundle E.

References

[1] Griffiths, P., & Harris, J. Principles of Algebraic Geometry. John Wiley & Sons.

[2] Harder, G., Narasimhan, M.S. On the cohomology groups of moduli spaces of vector bundles on curves. Math. Ann. 212, 215-248 (1975).
`https://doi.org/10.1007/BF01357141`

[3] Schaffhauser, F. (2013). Differential geometry of holomorphic vector bundles on a curve. Geometric and Topological Methods for Quantum Field Theory: Proceedings of the 2009 Villa de Leyva Summer School, 39.